MAPPING CLASS GROUP DYNAMICS

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1. Character varieties

In this introductory section we give the definitions and properties we need. We try to give heuristic and we refer to the references for the proofs. We use [Mar22b], [Sik12] and [Gol84] as our principal references.

1.1. **Representation varieties.**

Let Γ be a finitely generated group (in the following it will be a free group or a surface group). Let G be a matrix group (a closed subgroup of a linear group).

Example 1.1. We will only consider the so-called classical Lie groups, for example:

- $GL(d,k)$ is the group of linear automorphism of k^d, for $k = \mathbb{R}, \mathbb{C}$.
- $\mathrm{SL}(d,k)$ is the subgroup of $GL(k,\mathbb{C})$ of matrices with determinant 1.
- $SU(d) \subset \mathrm{SL}(d,\mathbb{C})$ is the subgroup of matrices preserving the standard Hermitian form on $\mathbb{C}^d$.

Definition 1. *The representation variety is the set* $\mathrm{Hom}(\Gamma, G)$ *of all morphisms (or representations) from* Γ *to* G*.*

As we might guess from its name, the representation variety is not merely a set. It can be given the structure of an algebraic (or analytic) variety. Let

$$\Gamma = \langle \gamma_1, \dots, \gamma_n \mid r \in R \rangle$$

be a presentation of Γ. This choice of generators gives an embedding

$$\begin{aligned} \mathrm{Hom}(\Gamma, G) &\to G^n \\ \rho &\mapsto (\rho(\gamma_1), \dots \rho(\gamma_n)). \end{aligned}$$

Each relation $r \in R$ is a word in the generators. It defines a map $r : G^n \to G$ whichs maps $(g_1, \dots, g_n)$ to the element of G obtained by replacing each γ_i by g_i in r. This map is polynomial in the matrix coefficients. The image of the embedding above is exactly the subset of G^n defined by the equations

$$r(g_1, \dots, g_n) = e,$$

for each $r \in R$. These equations are polynomial, so they define a subvariety of G^n (here G^n is given the structure of affine algebraic variety induced from the embedding of G in a linear group). Given another presentation of Γ the polynomial map which express a generator in one presentation as a word in the generators of the other presentation is an isomorphism between the two algebraic varieties defined by the two presentations. In summary:

Proposition 1.1. *A presentation of* Γ *induces a structure of algebraic variety on* $\mathrm{Hom}(\Gamma, G)$*. The structure of algebraic variety does not depend of the presentation (up to isomorphism).*

Proof. See [Mar22b], Lemma 1.2.2 and Lemma 1.2.3 . $\square$

Remark. The representation variety is a set of functions $\Gamma \to G$ so it can be given the topology of pointwise convergence. The embeding defined above is then a homeomorphism onto its image.

1.2. **Smooth points, Zariski tangent space.** We would like to determine the smooth points of the representation variety. A convenient way to do that is to determine its *Zariski tangent spaces.*

Definition 2. *Let $V \subset \mathbb{C}^N$ be a subvariety defined by the equation $f = 0$ where $f : \mathbb{C}^N \to \mathbb{C}^{N'}$ is a polynomial map. The Zariski tangent space T_xV of V at $x \in V$ is the kernel of the differential $D_xf : T_x\mathbb{C}^N \to T_x\mathbb{C}^{N'}$.*

If V is as above, observe that V is smooth at x if and only if the rank of D_xf is locally constant at x, which is equivalent to the dimension of T_xV being locally constant. In that case the Zariski tangent space is simply the usual tangent space.

We recall that the tangent space at the identity of G is by definition the Lie algebra $\mathfrak{g}$ of G. The exponential map $\exp : \mathfrak{g} \to G$ (given by the usual exponential of matrices) gives analytic paths $t \mapsto e^{tX}$ tangent to $X \in \mathfrak{g}$ at the identity. The tangent bundle of G is naturally trivialized in the following way:

$$\begin{aligned} G \times \mathfrak{g} &\to TG \\ (g, v) &\mapsto D_eR_gv, \end{aligned}$$

where $R_g : G \to G$ is the multiplication on the right by g. More concretely, at first order, a path at g is of the form $e^{tX}g$ where e^{tX} is a path at the identity tangent to $X \in \mathfrak{g}$.

Example 1.2. The Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ of $\mathrm{SL}(n, \mathbb{C})$ is the space of matrices with zero trace. The Lie algbera $\mathfrak{su}(n)$ of $SU(n)$ is the space of anti-Hermitian matrices.

To determine the Zariski tangent space of $\mathrm{Hom}(\Gamma, G)$ at ρ we use the following heuristic calculation. Let (ρ_t) be a path of representations such that $\rho_0 = \rho$. For each $\gamma \in \Gamma$ there exists $c(\gamma) \in \mathfrak{g}$ such that

$$\rho_t(\gamma) = e^{tc(\gamma)}\rho_0(\gamma) + o(t)$$

at first order. For each $\gamma_1, \gamma_2 \in \Gamma$ we have

$$\rho_t(\gamma_1\gamma_2) = \rho_t(\gamma_1)\rho_t(\gamma_2),$$

which at first order gives:

$$e^{tc(\gamma_1\gamma_2)}\rho(\gamma_1\gamma_2) = e^{tc(\gamma_1)}\rho(\gamma_1)e^{tc(\gamma_2)}\rho(\gamma_2) + o(t),$$

ie

$$c(\gamma_1\gamma_2)\rho(\gamma_1\gamma_2) = c(\gamma_1)\rho(\gamma_1)\rho(\gamma_2) + \rho(\gamma_1)c(\gamma_2)\rho(\gamma_2),$$

ie

$$c(\gamma_1\gamma_2) = c(\gamma_1) + \rho(\gamma_1)c(\gamma_2)\rho(\gamma_1)^{-1},$$

which can be rewritten

$$c(\gamma_1\gamma_2) = c(\gamma_1) + Ad_\rho(\gamma_1)c(\gamma_2),$$

where Ad_ρ is the adjoint action of Γ on $\mathfrak{g}$ given by ρ:

$$Ad_\rho(\gamma)X = \rho(\gamma)X\rho(\gamma)^{-1}.$$

The relation above means that the map $c : \Gamma \to \mathfrak{g}$ is a *1-cocycle* for the structure of Γ-module of $\mathfrak{g}$ (ie the Γ-action) given by Ad_ρ. This Γ-module is denoted by $\mathfrak{g}_\rho$.

Definition 3. *A map $c : \Gamma \to \mathfrak{g}_\rho$ is a 1-cocycle if it satisfies*

$$c(\gamma_1\gamma_2) = c(\gamma_1) + Ad_\rho(\gamma_1)c(\gamma_2)$$

for every $\gamma_1, \gamma_2 \in \Gamma$. The linear space of 1-cocycles is denoted by $Z^1(\Gamma, \mathfrak{g}_\rho)$.

The reasoning above can be formalized to give:

Proposition 1.2. *The Zariski tangent space of* $\mathrm{Hom}(\Gamma, G)$ *at ρ is the space of 1-cocycle $Z^1(\Gamma, \mathfrak{g}_\rho)$.*

Proof. See [Mar22b, Cor. 1.4.5] or [Gol84, §1.2]. For a scheme-theoretical point of view, see [Sik12, Th. 35]. □

Example 1.3. If $\Gamma = \langle a_1, \dots, a_r\rangle$ is a free group or rank r then $\mathrm{Hom}(\Gamma, G)$ is simply G^r, which is a smooth manifold. The Zariski tangent space at any representation ρ is $\mathfrak{g}^r$. The cocycle $c : \Gamma \to \mathfrak{g}_\rho$ associated to $(v_1, \dots, v_r) \in \mathfrak{g}^r$ is determined by $c(a_i) = v_i$ and the cocycle relations.

Determining the smooth points of the representation variety is hard in general. For a surface group we have the following criterion due to Goldman.

Proposition 1.3. *If Γ is a surface group of genus g and G is semisimple then the dimension of the Zariski tangent space of* $\mathrm{Hom}(\Gamma, G)$ *at ρ is*

$$\dim Z^1(\Gamma, \mathfrak{g}_\rho) = (2g - 1)\dim G + \dim C(\rho).$$

In particular, ρ is a smooth point exactly when $\dim C(\rho) = \dim C(G)$.

Proof. Let $r = \Pi[a_i, b_i]$ be the unique relator of the surface group. The idea is to show that the rank of the derivative $D_\rho r : \mathfrak{g}^{2g} \to \mathfrak{g}$ is $\dim G - \dim C(\rho)$.

It can be obtained by a calculation of the image of the derivative $D_\rho r$. See [Gol84, Prop. 1.2, Prop. 3.7] for a calculation using *Fox's free differential calculus*, or [Mar22b, Prop. 1.5.3] for a direct calculation.

There is also a *cohomological* interpretation of this formula, see next section. □

In the proposition above, $C(\rho)$ and $C(G)$ denote the centralizer of ρ and G and we say that G is semisimple when $(g, h) \mapsto \mathrm{tr}(gh)$ is a non-degenerate bilinear form on $\mathfrak{g}$ (it will be the case for all the groups we consider).

1.3. **Conjugation, character variety.** The group G acts on itself by conjugacy and this action extends to an action on $\mathrm{Hom}(\Gamma, G)$ by post-composition:

$$(g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1}.$$

This action is given polynomial automorphisms. The representation ρ and $g \cdot \rho$ have the same algebraic (geometrical, dynamical) properties and in many situations should be considered the same. We would like to consider the quotient of $\mathrm{Hom}(\Gamma, G)$ by this action, but several problems arise: the action might not be free and it might not be proper. Observe that $C(G)$ acts trivially and we denote by $Int(G)$ the group $G/C(G)$.

One radical solution is to take the *GIT quotient*. When a (reductive) algebraic group G acts on an affine algebraic variety V, there is a GIT (or categorical quotient) $V//G$ defined in the following way. Let $\mathbb{C}[V]$ be the algebra of regular functions on V. Denote by $\mathbb{C}[V]^G$ the subalgebra of functions invariant by the action of G (by precomposition). This algebra is finitely generated (ref). The quotient $V//G$ is by definition the affine variety associated to $\mathbb{C}[V]^G$ (*ie* the set of maximal ideals of $\mathbb{C}[V]^G$). There is a map $V \to V//G$ dual to the inclusion $\mathbb{C}[V]^G \subset \mathbb{C}[V]$. It is characterized by the following universal property: a regular map $V \to W$ is constant on G-orbits if and only if it factorizes through $V \to V//G$.

The discussion above applies to $V = \mathrm{Hom}(\Gamma, G)$. The GIT quotient $\mathrm{Hom}(\Gamma, G)//G$ is called the character variety and denoted by $\mathcal{X}(\Gamma, G)$. We have to be carefull that the "quotient map" $\pi : \mathrm{Hom}(\Gamma, G) \to \mathcal{X}(\Gamma, G)$ is not the set-theoretic quotient map: we may have $\pi(\rho) = \pi(\rho')$ but ρ is not conjugated to ρ'. In fact we have the following:

Proposition 1.4. *For $\rho, \rho' \in \mathrm{Hom}(\Gamma, G)$, we have $\pi(\rho) = \pi(\rho')$ if and only if*

$$\overline{G \cdot \rho} \cap \overline{G \cdot \rho'} \neq \emptyset.$$

In every fiber $\pi^{-1}([\rho])$ there exists a unique closed orbit.

Proof. This is a general property about GIT quotient. See for example [Mar22a, Sec. 3.2] for a readable introduction to GIT applied to character varieties. □

The GIT quotient can be tricky to work with. We look for a "smooth" model of this quotient. To do that, we analyze further the property of the G-action.

Proposition 1.5. *The stabilizer of ρ is $C(\rho)$, so the action of $Int(G)$ is free on the set of representations such that $C(G) = C(\rho)$ and is locally free at ρ if $\dim C(\rho) = \dim C(G)$ (ie if $C(\rho)/C(G)$ is discrete).*

Proof. It can be seen by an analysis of the orbit map $g \mapsto g \cdot \rho$. □

We say that ρ is *regular* if $\dim C(\rho) = \dim C(G)$ and *very regular* if $C(G) = C(\rho)$.

In the following we assume that $G = \mathrm{SL}(n, \mathbb{C})$ or $SU(n)$. A representation $\rho \in \mathrm{Hom}(\Gamma, G)$ is *irreducible* if ρ does not fix a proper subspace of $\mathbb{C}^n$. The set of irreducible representations is denoted by $\mathrm{Hom}^{irr}(\Gamma, G)$ This notion is fundamental because of the following proposition.

Proposition 1.6. *If $\rho \in \mathrm{Hom}(\Gamma, G)$ is irreducible then it is very regular and the orbit $G \cdot \rho$ is closed. The action of $Int(G)$ on $\mathrm{Hom}^{irr}(\Gamma, G)$ is proper. The set $\mathrm{Hom}^{irr}(\Gamma, G)$ is Zariski-open in $\mathrm{Hom}(\Gamma, G)$, and non-empty if Γ is a free group or a surface group.*

Proof. See [Mar22b, Prop. 2.2.9, Th. 2.2.10] and references therein or the discussion in [Gol84, §1.4]. □

Observe that when Γ is a surface group (or a free group), a very regular representation is a smooth point by the proposition 1.3. This gives:

Proposition 1.7. *If Γ is a free group or a surface group then the action of $Int(G)$ on $\mathrm{Hom}^{irr}(\Gamma, G)$ is free and proper. The quotient $\mathrm{Hom}^{irr}(\Gamma, G)/Int(G)$ is a smooth manifold. If Γ is a surface group of genus g then its dimension is*

$$(2g - 2) \dim G + 2 \dim C(G),$$

and if Γ is a free group of rank r then its dimension is

$$(r - 1) \dim G + \dim C(G).$$

Proof. See [Mar22b, Cor. 2.2.14] or [Sik12, Prop. 49, Cor. 50]. □

The quotient $\mathrm{Hom}^{irr}(\Gamma, G)/Int(G)$ is denoted by $\mathcal{X}^{irr}(\Gamma, G)$ and by the proposition 1.6 it is an open subset of $\mathcal{X}(\Gamma, G)$.

1.4. **Tangent space of the character variety.** We want to determine the tangent space to the character variety. Morally, the tangent space $T_\rho\mathcal{X}(\Gamma, G)$ should be $T_\rho\mathrm{Hom}(\Gamma, G)/T_\rho\,(G\cdot\rho)$. We start by finding $T_\rho\,(G\cdot\rho)$.

Let (ρ_t) be a path such that $\rho_0 = \rho$ and contained in $G\cdot\rho$. Then there exists a path (g_t) in G with $g_0 = e$ such that for every $\gamma \in \Gamma$:

$$\rho_t(\gamma) = g_t\rho(\gamma)g_t^{-1}.$$

At first order we have

$$g_t = e^{tX} + o(t),$$

so that

$$\rho_t(\gamma) = e^{tX}\rho(\gamma)e^{-tX} + o(t),$$

and

$$\rho_t(\gamma) = e^{tX}\rho(\gamma)e^{-tX} + o(t).$$

If $c : \Gamma \to \mathfrak{g}_\rho$ is the cocycle tangent to (ρ_t) we then have:

$$c(\gamma)\rho(\gamma) = X\rho(\gamma) - \rho(\gamma)X,$$

ie

$$c(\gamma) = X - \rho(\gamma)X\rho(\gamma)^{-1} = X - Ad_\rho(\gamma)X.$$

We say that the 1-cocycle defined as above is the *coboundary* of $X \in \mathfrak{g}_\rho$. The set of 1-coboundary is denoted by

$$B^1(\Gamma, \mathfrak{g}_\rho) = \left\{c \in Z^1(\Gamma, \mathfrak{g}_\rho) \mid \exists X \in \mathfrak{g}, \forall\gamma \in \Gamma, c(\gamma) = X - Ad_\rho(\gamma)X\right\}.$$

This formal calculation can be made precise:

Proposition 1.8. *The tangent space to the orbit*

$$T_\rho\,(G\cdot\rho) \subset T_\rho\mathrm{Hom}(\Gamma, G)$$

is the subspace

$$B^1(\Gamma, \mathfrak{g}_\rho) \subset Z^1(\Gamma, \mathfrak{g}_\rho).$$

Proof. See [Mar22b, Prop. 1.4.6] and [Sik12, Th. 43]. □

1.5. **A cohomological interlude.** The terms cocyle and coboundary come from the fact there exists a differential complex where they form actual cocycles and coboundaries, the so-called *bar resolution* of the twisted cohomology of Γ. We recall here the necessary definitions, refering to the standard reference [Bro82, p. I.5] or [Mar22b, Ap. B] (for a quick introduction). Let M be a Γ-module. Let $C^n(\Gamma, M)$ be the set of functions $\Gamma^n \to M$. An element $f \in C^n(\Gamma, M)$ is called a n-cochain. We define the differential $\delta : C^n(\Gamma, M) \to C^{n+1}(\Gamma, M)$:

$$\begin{aligned}\delta f(g_1, \ldots, g_{n+1}) = {} & g_1 f(g_2, \ldots, g_{n+1}) \\ & + \sum_{i=1}^{n}(-1)^i f(g_1, \ldots, g_{i-1}, g_ig_{i+1}, g_{i+2}, \ldots, g_{n+1}) \\ & + (-1)^{n+1} f(g_1, \ldots, g_n).\end{aligned}$$

We define the space of *n-cocycles*

$$B^n(\Gamma, M) = \ker\delta_{|C^n} \subset C^n(\Gamma, M),$$

and the space of *n-coboundaries*

$$Z^n(\Gamma, M) = \mathrm{Im}\delta_{|C^{n-1}} \subset C^n(\Gamma, M),$$

and, as usual,

$$H^n(\Gamma, M) = B^n(\Gamma, M)/Z^n(\Gamma, M).$$

This is the n-th cohomology group of Γ with twisted coefficients in M. Formally, this is the cohomology of the bar resolution of the trivial Γ-module $\mathbb{Z}$ to which we apply the functor $\hom_{\mathbb{Z}\Gamma}(\cdot, M)$.

For $n = 0$, a 0-cochain is simply a $v \in M$ and

$$\delta(v)(\gamma) = \gamma \cdot v - v,$$

so a 0-cocycle is is an element M fixed by every $\gamma \in \Gamma$. We showed that $H^0(\Gamma, M) = M^\Gamma$, the subspace of fixed elements.

For $n = 1$ a 1-cochain is a map $c : \Gamma \to M$ and

$$\delta c(\gamma_1, \gamma_2) = \gamma_1 c(\gamma_2) - c(\gamma_1\gamma_2) + c(\gamma_2),$$

so we see that the notions of cocycle and coboundary that we defined earlier coincide.

One interest of this cohomological point of view is that the cohomology groups can be computed using any free resolution. For example, if Γ is the fundamental group of a CW-complex X whose universal covering is contractible (*ie* X is a $K(\Gamma, 1)$) then $H^*(\Gamma, \mathbb{Z}) = H^*(X, \mathbb{Z})$, where $\mathbb{Z}$ is considered a trivial Γ-module.

1.5.1. *Twisted de Rham's cohomology.* When X is a smooth compact manifold and M is a finite dimensional vector space, we explain how to compute $H^*(\Gamma, M)$ using a version twisted version of de Rham's comohology.

We start by defining a vector bundle X_M on X. The fundamental group Γ acts diagonaly on the product $\widetilde{X} \times M$. The first projection $\widetilde{X} \times M \to \widetilde{X}$ is a vector bundle with fiber M and is equivariant for the Γ-action. If we define $X_M = (\widetilde{X} \times M)/\Gamma$ then the projection induces a vector bundle $X_M \to X$ with fiber M. This is the vector bundle associated to the Γ-module M.

This vector bundle is naturally equiped with an extra structure: a *flat connection.* We recall that a *connection* on a vector bundle $E \to N$ over a manifold N, is a map ∇ which maps a section s of E to a section ∇s of $TN^* \otimes E$ (*ie* a 1-form with values in E) and which satisfies Leibniz's rule:

$$\nabla(fs) = df \otimes s + f\nabla s.$$

A connection automatically induces maps $\nabla : \Omega^k(E) \to \Omega^{k+1}(E)$ (where $\Omega^k(E)$ is the space of k-forms with values in E), by imposing the rule:

$$\nabla(\alpha \otimes s) = d\alpha \otimes s + \alpha \wedge \nabla s,$$

where α is a k-form and s is a section of E. The connection ∇ is called *flat* when $\nabla^2 : \Omega^0(E) \to \Omega^2(E)$ is zero. This forces that every $\nabla^2 : \Omega^k(E) \to \Omega^{k+2}(E)$ is zero.

We go back to our vector bundle $X_M \to X$. At the level $\widetilde{X} \times M \to \widetilde{X}$, there is a connection defined by the product structure. Indeed, a section of this bundle is simply a map $s : \widetilde{X} \to M$ and we define ∇s to be ds (this is well-defined because s takes values in a vector space). This connection is flat because $d^2 = 0$. This connection commutes with the action of Γ so it induces a flat connection ∇ on the vector bundle $X_M \to X$.

We now have a complex

$$\Omega^0(X_M) \xrightarrow{\nabla} \Omega^1(X_M) \xrightarrow{\nabla} \dots \xrightarrow{\nabla} \Omega^k(X_M) \xrightarrow{\nabla} \dots$$

where $\nabla^2 = 0$. We can define, as usual, the cohomology groups $H^*(X, X_M)$:

$$H^k(X, X_M) = \frac{\ker \nabla_{|\Omega^k(X_M)}}{\mathrm{Im}\nabla_{|\Omega^{k-1}(X_M)}}.$$

We have the expected result:

Proposition 1.9. *If Γ is the fundamental group of a compact manifold X such that $\widetilde{X}$ is contractible and (X_M, ∇) is the associated flat vector bundle then*

$$H^*(\Gamma, M) \cong H^*(X, X_M).$$

This implies in particular a finiteness result: if the dimension of X is d, then $H^k(\Gamma, M) = 0$ for $k > d$, because there is no k-forms on X.

Let us check this result in degree 0. An element of $H^0(X, X_M)$ is a section s of X_M such that $\nabla s = 0$. The section $s : X \to X_M$ lifts to an equivariant map $\widetilde{s} : \widetilde{X} \to M$. The condition $\nabla s = 0$ means that $d\widetilde{s} = 0$, ie that $\widetilde{s}$ is constant. But as $\widetilde{s}$ is equivariant, this means that its unique value $v \in M$ is fixed by every $\gamma \in \Gamma$. Reciprocally, by any $v \in M$ fixed by every $\gamma \in \Gamma$ defines a constant equivariant map. So $H^0(X, X_M)$ is the space M^Γ of Γ-invariants, and we saw in the previous section that $H^0(\Gamma, M) = M^\Gamma$.

1.5.2. *Product.* The other interest is that the usual cohomological structures are available. For example, there exists a *cup product*:

$$\cup : H^p(\Gamma, M) \otimes H^q(\Gamma, N) \to H^{p+q}(\Gamma, M \otimes N),$$

generalizing the usual cup product, defined by:

$$(f \cup g)(\gamma_1, \dots, \gamma_{p+q}) = (-1)^{pq} f(\gamma_1, \dots, \gamma_p) \otimes \gamma_1 \cdots \gamma_p g(\gamma_{p+1}, \dots, \gamma_{p+q}),$$

at the level of cochains. Throught the isomorphim $H^*(\Gamma, M) \cong H^*(X, X_M)$ the cup product is the wedge product at the level of smooth forms:

$$\begin{aligned} \wedge : H^p(X, X_M) \otimes H^q(X, X_N) &\to H^{p+q}(X, X_{M\otimes N}) \\ (\alpha_1 \otimes s_1, \alpha_2 \otimes s_2) &\mapsto (\alpha_1 \wedge \alpha_2) \otimes (s_1 \otimes s_2). \end{aligned}$$

1.6. **Back to the tangent space.** We said earlier that the tangent space to the character variety $\mathcal{X}(\Gamma, G)$ should be $T_\rho \mathrm{Hom}(\Gamma, G)/T_\rho (G \cdot \rho)$. With the cohomological concept we introduced, this means that it should simply be $H^1(\Gamma, \mathfrak{g}_\rho)$. This is indeed the case, at least at the good points:

Proposition 1.10. *The tangent space at $[\rho] \in \mathcal{X}^{irr}(\Gamma, G)$ is $H^1(\Gamma, \mathfrak{g}_\rho)$.*

Proof. This follows from the description of the tangent space of $\mathrm{Hom}^{irr}(\Gamma, G)$ (Proposition 1.2), of the tangent space of the orbit (Proposition 1.8), and the fact that the action is free and proper on $\mathrm{Hom}^{irr}(\Gamma, G)$ (Proposition 1.7). ∎

1.7. **Symplectic form.** In the following, we assume that Γ is the fundamental group of a closed surface of genus $g \geq 2$ and that $G = \mathrm{SL}(d, \mathbb{C})$ or $G = SU(d)$.

In this section we define a symplectic form on $\mathrm{Hom}^{irr}(\Gamma, G)$. We recall that a symplectic form ω on a manifold M is a closed 2-form which is *nondegenerate:* for every $x \in M$, the bilinear pairing $\omega_x : T_xM \otimes T_xM \to \mathbb{R}$ is nondegenerate.

We denote by $B : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ the symmetric bilinear form on $\mathfrak{g}$ given by

$$B(X, Y) = \mathrm{Re}\,(\mathrm{tr} XY)\,.$$

The form B is nondegenerate and invariant by the adjoint action of G. This implies that the G-module $\mathfrak{g}$ is isomorphic to its dual $\mathfrak{g}^*$ as a G-module. In particular, for every $\rho \in \mathrm{Hom}(\Gamma, G)$ the Γ-module $\mathfrak{g}_\rho$ is isomorphic to its dual $\mathfrak{g}_\rho^*$ as a Γ-module.

We now define the symplectic form ω on $\mathcal{X}^{irr}(\Gamma, G)$. Let $\rho_0 \in \mathrm{Hom}^{irr}(\Gamma, G)$ and $[\rho_0]$ its image in $\mathcal{X}^{irr}(\Gamma, G)$. The tangent space to $\mathcal{X}^{irr}(\Gamma, G)$ at $[\rho_0]$ is $H^1(\Gamma, \mathfrak{g}_{\rho_0})$. Let $c_1, c_2 \in H^1(\Gamma, \mathfrak{g}_{\rho_0})$. We define the value $\omega_{[\rho_0]}(c_1, c_2)$ in 3 steps:

(1) As explained in the section 1.5 the cup product of c_1 and c_2 gives a cohomology class
$$c_1 \cup c_2 \in H^2(\Gamma, \mathfrak{g}_{\rho_0} \otimes \mathfrak{g}_{\rho_0}).$$
(2) Using the form $B : \mathfrak{g}_{\rho_0} \otimes \mathfrak{g}_{\rho_0} \to \mathbb{R}$, we get a map
$$B_* : H^2(\Gamma, \mathfrak{g}_{\rho_0} \otimes \mathfrak{g}_{\rho_0}) \to H^2(\Gamma, \mathbb{R})$$
by postcomposing the cocycles with B. We obtain a cohomology class
$$B_*(c_1 \cup c_2) \in H^2(\Gamma, \mathbb{R}).$$
(3) Because Γ is the fundamental group of a surface S of genus $g \geq 2$ and the universal cover of S is contractible, the cohomology group $H^2(\Gamma, \mathbb{R})$ is naturally isomorphic to $H^2(S, \mathbb{R})$. The fundamental class $[S] \in H_2(S, \mathbb{R})$ gives an isomorphism $\cap[S] : H^2(S, \mathbb{R}) \xrightarrow{\sim} \mathbb{R}$. We define $\omega_{[\rho_0]}(c_1, c_2)$ to be
$$B_*(c_1 \cup c_2) \cap [S] \in \mathbb{R}.$$

Theorem 1 (Goldman)**.** *The 2-form ω is a symplectic form on* $\mathrm{Hom}^{irr}(\Gamma, G)$*.*

Proof. See [Gol84, §1.7]. Each ω_{ρ_0} is alternating because the cup product $\cup$ is and B is symmetric. It is nondegenerate by *Poincaré duality* for group cohomology and the nondegenerateness of B.

The fact that ω is closed is proved in [Gol84, §1.7]. with an analytical point of view and in [Kar92, Th. 4] with an algebraic point of view. □

There is an "explicit" expression of ω which shows that it is *algebraic*, see [Gol84, §3.10]. It amounts to expliciting the fundamental class $[S]$ in group cohomology.

Henceforth, we equip $\mathrm{Hom}^{irr}(\Gamma, G)$ with this symplectic form. It is then a symplectic manifold.

A symplectic structure naturally gives us several associated object. In general, let (M, ω) be a symplectic manifold. We list some of the consequences:

(1) The dimension of M is even (by nondegenerateness of ω); let $\dim M = 2d$.
(2) The top-dimensional form ω^d is a volume form.
(3) A diffeomorphism of M preserving ω is called a *symplectomorphism*. Any symplectomorphism preserves the associated volume form ω^d.

(4) By nondegenerateness, for each $x \in M$ we have an isomorphism $T_xM \xrightarrow{\sim} T_xM^*$ given by $v \mapsto \omega_x(v, \cdot)$. So, for each vector vield X on M we have an associated 1-form $\omega(X, \cdot)$ and for each 1-form α on M there is a uniquely defined vector field X such that $\alpha = \omega(X, \cdot)$.

(5) For a smooth function f on M, we denote by X_f the vector field associated to the 1-form df. It can be thought of as a *symplectic gradient* It satisfies by definition $\omega(X_f, Y) = df(Y)$ for any vector field Y.

(6) For a smooth function f on M, the (non necessarly complete) flow (φ_t^f) generated by the vector field X_f is called the *Hamiltonian flow* associated to f. This is a flow of symplectomorphism. It is tangent to the fibers of f:
$$\partial_t f(\varphi_t^f) = df(X_f) = \omega(X_f, X_f) = 0.$$

(7) For smooth functions f and g we define their *Poisson bracket*:
$$\{f, g\} = \omega(X_f, X_g) = df(X_g) = -dg(X_g).$$
We say that f and g Poisson-commute if $\{f, g\} = 0$. This means that the Hamiltonian flow associated to g is contained in the fibers of f.

1.8. **Characters.** In this section we assume that $G = \mathrm{SL}(n, \mathbb{C})$. The *character* of a representation $\rho : \Gamma \to G$ is the function $\mathrm{tr}\rho : \Gamma \to \mathbb{C}$, $\gamma \mapsto \mathrm{tr}\rho(\gamma)$. We recall that in the setting of finite group, the character of representations determines the representation (up to isomorphism). In general this is false: the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (and all of their powers) have the same trace but are not conjugated. The next best thing is true:

Proposition 1.11. *The conjugacy class of an irreducible representation $\rho \in \mathrm{Hom}^{irr}(\Gamma, G)$ is determined by its character* $\mathrm{tr}\rho$.

In other words, the map
$$\begin{aligned} \mathcal{X}^{irr}(\Gamma, G) &\to \mathbb{C}^\Gamma \\ [\rho] &\mapsto \mathrm{tr}\rho \end{aligned}$$
is injective.

Proof. See [Gol09, Sec.2.3]. This follows from Burnside's lemma (if ρ is irreducible, the vector space spanned by the image of $\rho(\Gamma)$ is the space of all matrices $M_n(\mathbb{C})$) and the fact that any algebra automorphism of $M_n(\mathbb{C})$ is inner. □

A similar statement is true at the algebraic level. Observe that for each $\gamma \in \Gamma$, the regular function $\mathrm{tr}\gamma : \mathrm{Hom}(\Gamma, G) \to \mathbb{C}$, $\rho \mapsto \mathrm{tr}\rho(\gamma)$ is G-invariant and define a regular function on $\mathcal{X}(\Gamma, G)$.

Proposition 1.12. *The ring of regular functions $\mathbb{C}[\mathcal{X}(\Gamma, G)]$ on the character variety $\mathcal{X}(\Gamma, G)$ is generated by the trace functions $\mathrm{tr}\gamma$ for $\gamma \in \Gamma$. As this ring is finitely generated, there exists finitely many $\gamma_1, \ldots, \gamma_N \in \Gamma$ such that $\mathbb{C}[\mathcal{X}(\Gamma, G)] = \mathbb{C}[\mathrm{tr}\gamma_1, \ldots, \mathrm{tr}\gamma_N]$.*

This proposition means that any regular functions on $\mathcal{X}(\Gamma, G)$ is a polynomial in the functions $\mathrm{tr}\gamma_1, \ldots, \mathrm{tr}\gamma_N$. Going back to the definition of $\mathcal{X}(\Gamma, G)$, this means that any regular functions on $\mathrm{Hom}(\Gamma, G)$ invariant by conjugation is a polynomial in the functions $\mathrm{tr}\gamma_1, \ldots, \mathrm{tr}\gamma_N$.

Example 1.4. Let $\Gamma = \mathbb{Z}$. The representation variety $\mathrm{Hom}(\Gamma, G)$ is simply G and the character variety is $G//G$, the variety of closed conjugacy class in G. The conjugacy class of g is closed if and only if g is diagonalizable, as can be checked using Jordan's normal form. The conjugacy class of a diagonalizable matrix g is determined by its eigenvalues (with multiplicities), and those are determined by the traces $\mathrm{tr}g, \dots, \mathrm{tr}g^{d-1}$.

The proposition 1.12 imply that there is an injective regular map $\mathcal{X}(\Gamma, G) \to \mathbb{C}^N$ given by $\rho \mapsto (\mathrm{tr}\rho(\gamma_i))$. Be careful that in general there exists relations between the $\mathrm{tr}\gamma_i$.

1.9. **The free group on two generators.** In this section we consider the simplest non trivial case, and fundamental example of $\Gamma = \langle a, b \rangle$ the free group on 2 generators and $G = \mathrm{SL}(2, \mathbb{C})$.

Theorem 2 (Fricke)**.** *The map*

$$\begin{aligned} \mathcal{X}(\Gamma, G) &\to \mathbb{C}^3 \\ \rho &\mapsto (\mathrm{tr}\rho(a), \mathrm{tr}\rho(b), \mathrm{tr}\rho(ab)) \end{aligned}$$

is a regular isomorphism.

Proof. See [Gol09, Sec.2.2]. □

With the notation of the proposition 1.12 this means that the functions $\mathrm{tr}a, \mathrm{tr}b, \mathrm{tr}ab$ generate the ring of functions $\mathbb{C}\left[\mathcal{X}(\Gamma, G)\right]$ and that there is no relations between them.

For any word $w \in \Gamma$, the function $\mathrm{tr}w$ is defined on $\mathcal{X}(\Gamma, G)$, and by the proposition 1.12 and 2, is (in a unique way) a polynomial f_w in $\mathrm{tr}a, \mathrm{tr}b, \mathrm{tr}ab$. We often use the notation $x = \mathrm{tr}a, y = \mathrm{tr}b, z = \mathrm{tr}ab$. By using the *basic identity*

$$\mathrm{tr}XY + \mathrm{tr}XY^{-1} = \mathrm{tr}X\mathrm{tr}Y,$$

which is easily deduced from Cayley-Hamilton's theorem for $\mathrm{SL}(2, \mathbb{C})$, the polynomial f_w can be algorithmicaly found. An important example is the commutator $k = [a, b] = aba^{-1}b^1$. In this case we have

$$\kappa := f_k = x^2 + y^2 + z^2 - xyz - 2.$$

It can be checked that ρ is irreducible if and only if $\mathrm{tr}[\rho(a), \rho(b)] \neq 2$, *ie* if and only if $\kappa(x, y, z) \neq 2$. In particular, for $(x, y, z) \in \mathbb{C}^3$ if $\kappa(x, y, z) \neq 2$ then there exists a unique conjugacy class of $\rho \in \mathrm{Hom}(\Gamma, G)$ with these prescribed traces.

We can ask ourselves: what happens if the character (x, y, z) of a representation is in $\mathbb{R}$? is ρ necessarly conjugated to a representation in $\mathrm{SL}(2, \mathbb{R})$? the answer is no, not necessarly, but ρ is conjugated to a representations in a *real form* of $\mathrm{SL}(2, \mathbb{C})$.

Proposition 1.13. *Let $\rho \in \mathrm{Hom}(\Gamma, G)$ and suppose that its character (x, y, z) is in $\mathbb{R}^3$. Then either $x, y, z, \kappa(x, y, z) \in [-2, 2]$ and ρ is conjugated to a representations in $SU(2)$ or ρ is conjugated to a representations in* $\mathrm{SL}(2, \mathbb{R})$.

Proof. See [Gol09, Sec.3.4]. □

References

[Bro82] Kenneth S. Brown. *Cohomology of Groups*. Vol. 87. Graduate Texts in Mathematics. New York, NY: Springer New York, 1982. ISBN: 978-1-4684-9329-0 978-1-4684-9327-6. DOI: 10.1007/978-1-4684-9327-6.

[Gol09] William M. Goldman. "Trace Coordinates on Fricke Spaces of Some Simple Hyperbolic Surfaces". Jan. 10, 2009. arXiv: 0901.1404 [math]. URL: http://arxiv.org/abs/0901.1404 (visited on 04/27/2022).

[Gol84] William M Goldman. "The Symplectic Nature of Fundamental Groups of Surfaces". In: *Advances in Mathematics* 54.2 (Nov. 1984), pp. 200–225. ISSN: 00018708. DOI: 10.1016/0001-8708(84)90040-9.

[Kar92] Yael Karshon. "An Algebraic Proof for the Symplectic Structure of Moduli Space". In: *Proceedings of the American Mathematical Society* 116.3 (1992), pp. 591–605. ISSN: 0002-9939. DOI: 10.2307/2159424. JSTOR: 2159424.

[Mar22a] Julien Marché. "Character Variety". 2022. URL: https://webusers.imj-prg.fr/~julien.marche/papiers/Character_variety.pdf. preprint.

[Mar22b] Arnaud Maret. "A Note on Character Varieties". 2022. URL: https://arnaudmaret.com/files/character-varieties.pdf. preprint.

[Sik12] Adam Sikora. "Character Varieties". In: *Transactions of the American Mathematical Society* 364.10 (2012), pp. 5173–5208. ISSN: 0002-9947, 1088-6850. DOI: 10.1090/S0002-9947-2012-05448-1.